

Multi-scaling Limits for Relativistic Diffusion Equations with Random Initial Data

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Abstract

Let $u(t, \mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, be the spatial-temporal random field arising from the solution of a relativistic diffusion equation with the spatial-fractional parameter $\alpha \in (0, 2)$ and the mass parameter $\mathfrak{m} > 0$, subject to a random initial condition $u(0, \mathbf{x})$ which is characterized as a subordinated Gaussian field. In this article, we study the large-scale and the small-scale limits for the suitable space-time rescalings of the solution field $u(t, \mathbf{x})$. Both the Gaussian and the non-Gaussian limit theorems are discussed. The small-scale scaling involves not only to scale on $u(t, \mathbf{x})$ but also to re-scale the initial data; this is a new-type result for the literature. Moreover, in the two scalings the parameter $\alpha \in (0, 2)$ and the parameter $\mathfrak{m} > 0$ play distinct roles for the scaling and the limiting procedures.

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1 Introduction

In this paper, we consider the following Cauchy problem for the relativistic diffusion equation (RDE for brevity), *subject to some random initial data*, and aim to discuss the scaling limits for the spatial-temporal random field arising from the solution of this random initial value problem :

$$\frac{\partial}{\partial t}u(t, \mathbf{x}) = (\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}})u(t, \mathbf{x}), \quad u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad t \geq 0, \quad \mathbf{x} \in \mathbb{R}^n, \quad (1.1)$$

with the spatial-fractional parameter $\alpha \in (0, 2)$ and the (normalized) mass parameter $\mathfrak{m} > 0$.

RDEs appear in vast literature of mathematics and physics. The prominent case is $\alpha = 1$, for which $-(\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \Delta})$ is regarded as the free energy of the relativistic Schrödinger operator with a particle of mass \mathfrak{m} ; see the seminal paper of Carmona *et al.* [8] for mathematical discussions and its relation to Lévy processes. For general $\alpha \in (0, 2)$, one may refer to Ryznar [27], Baeumer *et al.* [4], Kumara *et al.* [17], and the references therein. RDEs have also played an essential role in the theory of computer vision; see a special volume edited by Kimmel *et al.* [15], in which P.D.E. and scale-space methods are focused and RDEs are particularly employed.

In this article, we consider the random initial data u_0 to be subordinated Gaussian random fields and study the large-scale and the small-scale limits for the properly re-scaled solution field. We prove that the two parameters α and $\mathfrak{m} > 0$ play distinct roles in the two scaling behaviors. For the large-scale limit (Theorem 1 and Theorem 3), it is the mass $\mathfrak{m} > 0$ dominates the space-time scaling and also the limiting field, which brings the $\mathfrak{m} > 0$ in its structure. While for the small-scale limit (Theorem 2 and Theorem 4), it is the spatial index α dominates both the scaling factor and the limiting field, and it appears to be irrelevant for \mathfrak{m} being positive or zero.

In our discussions, the large-scale Theorem 1 and Theorem 3 are respectively comparable to the Central Limit Theorem for local functionals of random fields with weak dependence in [7], and to a certain non-Gaussian Central Limit Theorem for which the

papers [29, 10] are pioneering. For the small-scale Theorem 2 and Theorem 4, they involve not only the space-time scaling on $u(t, \mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, but also need to re-scale the initial data; to our knowledge, these are new type results for the literature; see [22] for the authors' very recent study. As for the methodology for proofs, for the Gaussian limits we employ the moments and the Feymann-type diagrams used notably in [7], and for the non-Gaussian limits we employ the truncation of Hermite expansions used notably in [1, 2].

We remark that, in the non-relativistic case, i.e. $\mathbf{m} = 0$, the large-scale limits for the random initial value problem with multiple Itô-Wiener integrals as input have been discussed in Anh and Leonenko [1, 2]; subsequent works, together with Burgers' equation, in this direction by the authors and collaborators can be seen in [3, 5, 13, 19, 20, 21, 26] and the references therein. However, the multi-scaling limits due to the different roles of the mass and the fractional-index, the target of this article, are *at all not* in the cited papers. Moreover, in this article we are able to drop-off the usually imposed isotropic assumption of the initial datum.

We should also mention that, in an article discussing tempered stable Lévy processes by Rosiński [25], the author proves rigorously, among others, the statement that such a process in a short time looks a stable process while in a large time scale it looks like a Brownian motion. This article has surfaced nicely how the multi-scaling limits appear in the context of stochastic processes (We are indebted to the referee for indicating to us the article [25] and the relevant concept).

In Section 2, we present some preliminaries; we state our main results in Section 3, and all the proofs of our results are given in Section 4.

Finally, we mention that the study on the PDEs with random initial conditions can be traced back to [14] and [24]. Besides the above mentioned literature, there also has very significant progress on Burgers equation with different types of random input; see the monograph of Woyczyński [32] and the Chapter 6 of Bertoin [6].

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2 Preliminaries

2.1 Green function for RDEs

As understood, we regard the spatial operator in the RDE (1.1) as a pseudo-differential operator, see for example the book and the paper by Wong [31, 30]; the Green function, denoted by $G_{\alpha, \mathbf{m}}(t, \mathbf{x})$, $t > 0, \mathbf{x} \in \mathbb{R}^n$, for the Cauchy problem (1.1) is thus determined by the (spatial) Fourier transform $\widehat{G}_{\alpha, \mathbf{m}}(t, \lambda)$, $\alpha \in (0, 2)$, $\mathbf{m} > 0$, which is given by

$$\int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x} \rangle} G_{\alpha, \mathbf{m}}(t, \mathbf{x}) d\mathbf{x} = e^{-t\{(\mathbf{m}^{\frac{2}{\alpha}} + |\lambda|^2)^{\frac{\alpha}{2}} - \mathbf{m}\}}, \quad \lambda \in \mathbb{R}^n. \quad (2.1)$$

See Carmona *et al.* [8] for $\alpha = 1$ and Ryznar [27] for general $\alpha \in (0, 2)$ ([27] also considers the boundary problem). These papers also study $G_{\alpha, \mathbf{m}}(t, \mathbf{x})$, $\mathbf{m} > 0$, as the transition probability density of a Lèvy process $X_{\alpha, \mathbf{m}}(t)$ which is the subordination of the Brownian motion by a certain subordinator. The explicit expression for the Green function is known only in the case $\alpha = 1$; see for example the recent works of [4, 17], which give explicit calculations to show that the subordinator is normal inverse Gaussian.

The solution of (1.1) is given in the form

$$u(t, \mathbf{x}; u_0(\cdot)) = \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(t, \mathbf{x} - \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y}. \quad (2.2)$$

In this work, our initial data is a second-order homogeneous random field on \mathbb{R}^n , and thus the solution of (1.1) should be understood as a mean-square solution; resulting a spatial-temporal random solution field $u(t, \mathbf{x})$; see [26, Proposition 1] for some discussions on the mean-square solutions of parabolic PDEs with mean-square random initial data.

2.2 Subordinated Gaussian fields as initial data

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be an underlying probability space, such that all random elements appeared in this article are measurable with respect to it. We specify the initial data $u_0(\mathbf{x})$ be a subordinated Gaussian field, which is introduced by Dobrushin [9], as follows; see also [1, 2] for more recent discussions.

Condition A. The initial data of (1.1) is assumed to be a random field on \mathbb{R}^n given by

$$u_0(\mathbf{x}) = h(\zeta(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^n, \quad (2.3)$$

where $\zeta(\mathbf{x})$ is a mean-square continuous and homogeneous Gaussian random field with mean zero and variance 1, and its spectral measure $F(d\lambda)$ has the (spectral) density $f(\lambda)$, $\lambda \in \mathbb{R}^n$; moreover, $h : \mathbb{R} \rightarrow \mathbb{R}$ is a (non-random) function such that

$$\mathbb{E}h^2(\zeta(\mathbf{0})) = \int_{\mathbb{R}} h^2(r)p(r)dr < \infty; \quad p(r) = \frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}}, \quad r \in \mathbb{R}. \quad (2.4)$$

Under Condition A, by the Bochner-Khintchine theorem, we have the following spectral representation for the covariance function of the Gaussian field $\zeta(\mathbf{x})$:

$$R(\mathbf{x}) = \text{Cov}(\zeta(\mathbf{0}), \zeta(\mathbf{x})) = \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x} \rangle} f(\lambda) d\lambda. \quad (2.5)$$

Moreover, by the Karhunen Theorem, $\zeta(\mathbf{x})$ has the representation

$$\zeta(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x} \rangle} \sqrt{f(\lambda)} W(d\lambda), \quad \mathbf{x} \in \mathbb{R}^n, \quad (2.6)$$

where $W(d\lambda)$ is the standard complex-valued Gaussian white noise on the Fourier domain \mathbb{R}^n ; that is, a centered orthogonal-scattered Gaussian random measure on \mathbb{R}^n such that $W(\Delta_1) = \overline{W(-\Delta_1)}$ and $\mathbb{E}W(\Delta_1)\overline{W(\Delta_2)} = \text{Leb}(\Delta_1 \cap \Delta_2)$ for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}^n)$. See, for example, the book of Leonenko [18, Theorem 1.1.3] for the above facts. We need the following expansion of $h(r)$ in the Hilbert space $L^2(\mathbb{R}, p(r)dr)$:

$$h(r) = C_0 + \sum_{l=1}^{\infty} C_l \frac{H_l(r)}{\sqrt{l!}}, \quad (2.7)$$

where

$$C_l = \int_{\mathbb{R}} h(r) \frac{H_l(r)}{\sqrt{l!}} p(r) dr, \quad (2.8)$$

and $\{H_l(r), l = 0, 1, 2, \dots\}$ are the Hermite polynomials, that is,

$$H_l(r) = (-1)^l e^{\frac{r^2}{2}} \frac{d^l}{dr^l} e^{-\frac{r^2}{2}}, \quad \text{for } l \in \{0, 1, 2, \dots\}.$$

Accordingly, the *Hermite rank* of the function $h(\cdot)$ is defined by

$$m := \inf\{l \geq 1 : C_l \neq 0\}.$$

It is well-known that (see, for example, Major [23, Corollary 5.5 and p. 30]):

$$\mathbb{E}[H_{l_1}(\zeta(\mathbf{y}))H_{l_2}(\zeta(\mathbf{z}))] = \delta_{l_2}^{l_1} l_1! R^{l_1}(\mathbf{y} - \mathbf{z}), \quad \mathbf{y}, \mathbf{z} \in \mathbb{R}^n, \quad (2.9)$$

($\delta_{\sigma_2}^{\sigma_1}$ is the Kronecker symbol) and

$$H_l(\zeta(\mathbf{x})) = \int'_{\mathbb{R}^{n \times l}} e^{i\langle \mathbf{x}, \lambda_1 + \dots + \lambda_l \rangle} \prod_{k=1}^l \sqrt{f(\lambda_k)} W(d\lambda_k). \quad (2.10)$$

In the above, (2.10) means the *multiple Itô-Wiener integral* representation and the integration \int' means that it excludes the diagonal hyperplanes $\mathbf{z}_i = \mp \mathbf{z}_j$, $i, j = 1, \dots, l, i \neq j$.

We impose two different conditions on the singularity of the spectral density $f(\lambda)$ at $\mathbf{0}$, which yield, respectively, the Gaussian and the non-Gaussian scaling-limits.

Condition B. The spectral density function $f(\lambda)$ of the Gaussian random field $\zeta(\mathbf{x})$ in Condition A can be written as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}} \text{ for some } \kappa > \frac{n}{m}, \quad (2.11)$$

where m is the Hermite rank of the function h , and the $B(\cdot) \in C(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$.

Condition C. The spectral density function $f(\lambda)$ of the Gaussian random field $\zeta(\mathbf{x})$ in Condition A can be written as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}}, \quad 0 < \kappa < \frac{n}{m}, \quad (2.12)$$

where m is the Hermite rank of the function h , and the $B(\cdot) \in C(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$, and moreover $B(\mathbf{0}) > 0$.

Note that, in the two conditions, we do not assume that the $B(\cdot)$ is radial in \cdot , so that the field $u_0(\mathbf{x})$ is not necessary to be isotropic. We also mention that, the Condition B means that the density f either is regular at $\mathbf{0}$, or has a singularity for which the order is less than $n(1 - 1/m)$; while the Condition C means that f has a singularity at $\mathbf{0}$ for which the order is higher than $n(1 - 1/m)$.

By (2.5) and the convolutions, we have, for each $l \geq 1$,

$$R^l(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x} \rangle} f^{*l}(\lambda) d\lambda, \quad l \in \mathbb{N}, \quad (2.13)$$

where $f^{*l}(\lambda)$ is the l -fold convolution of f defined recursively as: $f^{*1} = f$ and

$$f^{*l}(\lambda) = \int_{\mathbb{R}^n} f(\lambda - \eta) f^{*(l-1)}(\eta) d\eta, \quad l \geq 2.$$

The following analytic lemma asserts the behavior of f^{*l} , $l \in \mathbb{N}$; for completeness, we give its proof in Appendix A.

Lemma 1. *Suppose that the spectral density function f has the form,*

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}}, \quad \kappa > 0,$$

*for some non-negative bounded and continuous function $B(\lambda)$ so that $f \in L^1(\mathbb{R}^n)$. Then for any $k \geq 2$ there exists a bounded function $B_k \in C(\mathbb{R}^n \setminus \{\mathbf{0}\})$ such that the k -fold convolution f^{*k} of f can be written as*

$$f^{*k}(\lambda) = \begin{cases} B_k(\lambda) |\lambda|^{k\kappa-n}, & \text{for } k\kappa < n, \\ B_k(\lambda) \ln(2 + \frac{1}{|\lambda|}), & \text{for } k\kappa = n, \\ B_k(\lambda) \in C(\mathbb{R}^n), & \text{for } k\kappa > n. \end{cases} \quad (2.14)$$

Moreover, for any $k_1 > k_2 > n/\kappa$ the inequality $\sup_{\lambda \in \mathbb{R}^n} B_{k_1}(\lambda) \leq \sup_{\lambda \in \mathbb{R}^n} B_{k_2}(\lambda)$ holds.

To understand the difference of the Conditions B and C, in view of Lemma 1, the Condition B implies that the k -fold convolution f^{*k} , $k \geq m$, has no singularity at the

origin $\lambda = \mathbf{0}$, which in turn asserts that the spectral density of the random initial data u_0 has no singularity at $\lambda = \mathbf{0}$; while the Condition C asserts that the initial data u_0 has a spectral density which is singularity at $\lambda = \mathbf{0}$. The situation can be described as, respectively, the long-range and the short-range dependence of the initial field u_0 ; a central notion in vast applications, as one may refer to the special volume by Doukhan, Oppenheim, and Taqqu [11].

3 Main results

The significant difference between the Condition B and the Condition C, as remarked at the end of the last section, is employed to obtain the Gaussian and respectively the non-Gaussian scaling-limits. We will present them in two subsections.

In the context henceforth, the notation \Rightarrow denotes the convergence of random variables (respectively, random families) in the sense of distribution (respectively, finite-dimensional distributions).

3.1 Gaussian limits with initial data in (A,B)

As mentioned in the Section 1, we will present the large-scale and the small-scale limit theorems. We remark that our Theorems 1 and 2 in this subsection are comparable to the central limit theorem for local functionals of random fields with weak dependence in Breuer and Major [7]. The novel feature is that the mass $\mathbf{m} > 0$ and the fractional-index α play different roles in the two-scales.

Theorem 1. *Let $u(t, \mathbf{x}; u_0(\cdot))$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, be the mean-square solution of (1.1) with $\mathbf{m} > 0$ and the initial data $u_0(\mathbf{x}) = h(\zeta(\mathbf{x}))$ satisfy Condition A and B with the Hermite rank $m \geq 1$. Then when $T \rightarrow \infty$,*

$$T^{\frac{n}{4}} \left\{ u(Tt, \sqrt{T}\mathbf{x}; u_0(\cdot)) - C_0 \right\} \Rightarrow U(t, \mathbf{x}),$$

where $U(t, \mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, is a Gaussian field with the following spectral representation:

$$U(t, \mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x} \rangle} \sigma_m e^{-t \frac{\alpha}{2} \mathbf{m}^1 - \frac{2}{\alpha} |\lambda|^2} W(d\lambda), \quad \sigma_m = \left(\sum_{r=m}^{\infty} f^{*r}(\mathbf{0}) C_r^2 \right)^{\frac{1}{2}}, \quad (3.1)$$

where $W(d\lambda)$ is a complex-valued standard Gaussian noise measure on \mathbb{R}^n (c.f. (2.6)).

For the small-scale limit, we need to re-scale the initial data too; thus the notation $u_0(\varepsilon^{-\frac{1}{\alpha}-\chi} \cdot)$ imposed on u_0 wants to mean that the variable of u_0 is under the indicated dilation factor $\varepsilon^{-\frac{1}{\alpha}-\chi}$.

Theorem 2. Let $u(t, \mathbf{x}; u_0(\cdot))$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, be the mean-square solution of (1.1) with $\mathbf{m} > 0$ and the initial data $u_0(\mathbf{x}) = h(\zeta(\mathbf{x}))$ satisfy Condition A and B with the Hermite rank $m \geq 1$. For any $\chi > 0$, when $\varepsilon \rightarrow 0$,

$$\varepsilon^{-\frac{n\chi}{2}} \left\{ u(\varepsilon t, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}; u_0(\varepsilon^{-\frac{1}{\alpha}-\chi} \cdot)) - C_0 \right\} \Rightarrow V(t, \mathbf{x}), \quad (3.2)$$

where $V(t, \mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, is a Gaussian field with the following spectral representation:

$$V(t, \mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x} \rangle} \sigma_m e^{-t|\lambda|^\alpha} W(d\lambda), \quad \sigma_m = \left(\sum_{r=m}^{\infty} f^{*r}(\mathbf{0}) C_r^2 \right)^{\frac{1}{2}}, \quad (3.3)$$

where $W(d\lambda)$ is a complex-valued standard Gaussian noise measure on \mathbb{R}^n .

Remark. The typical case for Theorem 2 is $\alpha = 1, \chi = 1/2$. In this critical case, the scaling order for Theorems 1 and 2 is the same, namely $n/4$. However, the spatial scaling is square-root in Theorem 1 while is linear in Theorem 2; moreover, the integral kernel for the limiting field in two theorems is Gauss vs. Poisson. The latter situation can be conferred to an analytic discussion in Wong [30].

3.2 Non-Gaussian limits with initial data in (A,C)

As in the above subsection, we have the large-scale and the small-scale limits; however the high singularity order in the Condition C assets that our limiting fields are now non-Gaussian. The non-Gaussian limits of the convolution type; which can be seen in

the pioneering papers of Taqqu [29] and Dobrushin and Major [10], and more recent Anh and Leonenko [1, 2].

Theorem 3. *Let $u(t, \mathbf{x}; u_0(\cdot))$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, be the mean-square solution of (1.1) whose initial data $\{u_0(\mathbf{x}) = h(\zeta(\mathbf{x})), \mathbf{x} \in \mathbb{R}^n\}$ satisfy Condition A and C with $\kappa \in (0, \frac{n}{m})$ and $1 < m$, where m is the Hermite rank of the non-random function h on \mathbb{R} , which has the Hermite coefficients $C_j, j = 0, 1, \dots$. Then when $T \rightarrow \infty$,*

$$T^{\frac{m\kappa}{4}} \left\{ u(Tt, \sqrt{T}\mathbf{x}; h(\zeta(\cdot))) - C_0 \right\} \Rightarrow U_m(t, \mathbf{x}), \quad (3.4)$$

where $U_m(t, \mathbf{x})$ is represented by the following multiple Wiener integrals

$$U_m(t, \mathbf{x}) = B^{\frac{m}{2}}(\mathbf{0}) \frac{C_m}{\sqrt{m!}} \int'_{\mathbb{R}^{n \times m}} e^{i\langle \mathbf{x}, \lambda_1 + \dots + \lambda_m \rangle} \frac{\exp(-t \frac{\alpha}{2} |\lambda_1 + \dots + \lambda_m|^2)}{(|\lambda_1| \dots |\lambda_m|)^{\frac{n-\kappa}{2}}} \prod_{l=1}^m W(d\lambda_l), \quad (3.5)$$

where $\int'_{\mathbb{R}^{n \times m}} \dots$ denotes an m -fold Wiener integral with respect to the complex Gaussian white noise $W(\cdot)$ on \mathbb{R}^n .

Theorem 4. *Let $u(t, \mathbf{x}; u_0(\cdot))$ be the mean-square solution to (1.1) whose initial data $\{u_0(\mathbf{x}) = h(\zeta(\mathbf{x})), \mathbf{x} \in \mathbb{R}^n\}$ satisfy Condition A and C with $\kappa \in (0, \frac{n}{m})$ and $1 < m$, where m is the Hermite rank of the function h . Then, for any fixed parameter $\chi > 0$, when $\varepsilon \rightarrow 0$,*

$$\varepsilon^{-\frac{m\kappa\chi}{2}} \left\{ u(\varepsilon t, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}; h(\zeta((\varepsilon^{-\frac{1}{\alpha}-\chi} \cdot))) - C_0 \right\} \Rightarrow V_m(t, \mathbf{x}), \quad (3.6)$$

where $V_m(t, \mathbf{x})$ is represented by the multiple Wiener integrals

$$V_m(t, \mathbf{x}) = B^{\frac{m}{2}}(\mathbf{0}) \frac{C_m}{\sqrt{m!}} \int'_{\mathbb{R}^{n \times m}} e^{i\langle \mathbf{x}, \lambda_1 + \dots + \lambda_m \rangle} \frac{\exp(-t |\lambda_1 + \dots + \lambda_m|^\alpha)}{(|\lambda_1| \dots |\lambda_m|)^{\frac{n-\kappa}{2}}} \prod_{l=1}^m W(d\lambda_l). \quad (3.7)$$

Remark. In [2] the authors considered a hybrid differential operator in the spatial variable (the Riesz-Bessel operator), as follows

$$-(-\Delta)^{\alpha/2} (I - \Delta)^{\gamma/2}, \quad \alpha \in (0, 2), \quad \gamma \geq 0.$$

However, in their main Theorem 2.3, a large-scale limit in our context, only the Riesz parameter α plays the role and the Bessel parameter γ is invisible. This intrigue situation

is now justified by the RFD (1.1), which we could say that it is “physically correct” to consider the relativistic operator $(\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}})$ rather than the Bessel operator in the form presented in [2].

4 Proofs of Theorems

The following two-scale property of the relativistic Green function $G_{\alpha, \mathfrak{m}}$ is the key to our results; when one deals the Laplacian or the fractional-Laplacian operator, it is instead only the mono-scaling. We describe this two-scale property in terms of Fourier transforms.

$$\widehat{G}_{\alpha, \mathfrak{m}}(Tt, T^{-\frac{1}{2}}\lambda) = \exp\left\{Tt(\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + T^{-1}|\lambda|^2)^{\frac{\alpha}{2}})\right\} \rightarrow \exp\left\{-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2\right\}, \quad (4.1)$$

as $T \rightarrow \infty$; (4.1) is a consequence of the of Taylor’s expansion,

$$\begin{aligned} \mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + T^{-1}|\lambda|^2)^{\frac{\alpha}{2}} &= \mathfrak{m} - \left(\mathfrak{m} + \frac{\alpha}{2}(\mathfrak{m}^{\frac{2}{\alpha}})^{\frac{\alpha}{2}-1}T^{-1}|\lambda|^2 + \frac{\alpha}{4}\left(\frac{\alpha}{2} - 1\right)c_T^{\frac{\alpha}{2}-2}T^{-2}|\lambda|^4\right) \\ &= -\frac{\alpha}{2}(\mathfrak{m}^{\frac{2}{\alpha}})^{\frac{\alpha}{2}-1}T^{-1}|\lambda|^2 + \frac{\alpha}{4}\left(1 - \frac{\alpha}{2}\right)c_T^{\frac{\alpha}{2}-2}T^{-2}|\lambda|^4 \end{aligned}$$

for some $c_T \in (\mathfrak{m}^{\frac{2}{\alpha}}, \mathfrak{m}^{\frac{2}{\alpha}} + T^{-1}|\lambda|^2)$. In contrast to the large-scale (4.1), we have the following small-scale, as $\varepsilon \rightarrow 0$,

$$\widehat{G}_{\alpha, \mathfrak{m}}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}}\lambda) = e^{\varepsilon t \mathfrak{m}} e^{-\varepsilon t(\mathfrak{m}^{\frac{2}{\alpha}} + \varepsilon^{-\frac{2}{\alpha}}|\lambda|^2)^{\frac{\alpha}{2}}} \rightarrow e^{-t|\lambda|^\alpha}. \quad (4.2)$$

We observe that (4.2) indeed holds no matter \mathfrak{m} is > 0 or $= 0$.

Proofs of Theorems 1 and 2.

We apply the Hermite expansion (2.7) to $u(t, x)$, For the large-scale, we set

$$\begin{aligned} X_T(t, \mathbf{x}) &= T^{\frac{n}{4}}u(Tt, \sqrt{T}\mathbf{x}; u_0(\cdot)) - C_0 \\ &= T^{\frac{n}{4}}\sum_{k=m}^{\infty}\frac{C_k}{\sqrt{k!}}\int_{\mathbb{R}^n}G_{\alpha, \mathfrak{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y})H_k(\zeta(\mathbf{y}))d\mathbf{y}, \end{aligned}$$

and for the small-scale, we set

$$\begin{aligned} Y_\varepsilon(t, \mathbf{x}) &:= \varepsilon^{-\frac{n\chi}{2}} u(\varepsilon t, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}; u_0(\varepsilon^{-\frac{1}{\alpha}-\chi} \cdot)) - C_0 \\ &= \varepsilon^{-\frac{n\chi}{2}} \sum_{l=m}^{\infty} \frac{C_l}{\sqrt{l!}} \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(\varepsilon t, \varepsilon^{\frac{1}{\alpha}} \mathbf{x} - \mathbf{y}) H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi} \mathbf{y})) d\mathbf{y}. \end{aligned}$$

In the below, we only proceed the proof of Theorem 2, the small-scale limit, and see how the rescaling of the initial data is needed to obtain the desired limit; the proof of Theorem 1 is parallel, and does not require the rescaling of the initial data. Since the proof in the following does not require the \mathbf{m} to be strictly positive, our Theorem 2 also provides a small-scale version of the large-scale, i.e. the usual, limit result in [2]. The methodology of the proof can be traced back to [7].

For any $M \in \mathbb{N}$ and any set of real numbers $\{a_1, a_2, \dots, a_M\}$, denote

$$\xi_\varepsilon := \sum_{j=1}^M a_j Y_\varepsilon(t_j, \mathbf{x}_j), \quad (4.3)$$

where $\{t_1, \dots, t_M\} \subset \mathbb{R}_+$ and $\{\mathbf{x}_1, \dots, \mathbf{x}_M\} \subset \mathbb{R}^n$ are arbitrary. In order to apply the Method of Moments to prove the statement of Theorem 2, we need to verify the following:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \xi_\varepsilon^p = \begin{cases} 0, & p = 2\nu + 1 \\ (p-1)!! \left\{ \mathbb{E} \left[\left(\sum_{j=1}^M a_j V(t_j, \mathbf{x}_j) \right)^2 \right] \right\}^\nu, & p = 2\nu \end{cases}, \quad (4.4)$$

where $V(t, \mathbf{x})$ is defined in (3.3). We remark that the high (i.e. $p > 2$) moments is needed, since ξ_ε is not Gaussian, though the wanted limit is Gaussian. Firstly, we split ξ_ε into two parts:

$$\xi_\varepsilon = \xi_{\varepsilon, \leq N} + \xi_{\varepsilon, > N}, \quad (4.5)$$

where (henceforth, we will suppress the indices α and \mathbf{m} for $G_{\alpha, \mathbf{m}}$ and $\widehat{G}_{\alpha, \mathbf{m}}$)

$$\xi_{\varepsilon, > N} = \sum_{j=1}^M a_j \varepsilon^{-\frac{n\chi}{2}} \sum_{l=N+1}^{\infty} \frac{C_l}{\sqrt{l!}} \int_{\mathbb{R}^n} G(\varepsilon t_j, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_j - \mathbf{y}) H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi} \mathbf{y})) d\mathbf{y}, \quad (4.6)$$

and we prove that $E[\xi_{\varepsilon, > N}^2] \rightarrow 0$, whenever N is chosen large enough. Observe that for any $N \geq m - 1$, by (2.9),

$$\begin{aligned} \mathbb{E}(\xi_{\varepsilon, > N})^2 &= \mathbb{E}\left[\left(\sum_{j=1}^M a_j \varepsilon^{-\frac{n\chi}{2}} \sum_{l=N+1}^{\infty} \frac{C_l}{\sqrt{l!}} \int_{\mathbb{R}^n} G(\varepsilon t_j, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_j - \mathbf{y}) H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi} \mathbf{y})) d\mathbf{y}\right)^2\right] \\ &= \sum_{j_1, j_2=1}^M a_{j_1} a_{j_2} \varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^{2n}} G(\varepsilon t_{j_1}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_1} - \mathbf{y}_1) G(\varepsilon t_{j_2}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_2} - \mathbf{y}_2) R^l(\varepsilon^{-\frac{1}{\alpha}-\chi}(\mathbf{y}_1 - \mathbf{y}_2)) \\ &= \sum_{j_1, j_2=1}^M a_{j_1} a_{j_2} \varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} G(\varepsilon(t_{j_1} + t_{j_2}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) - \mathbf{z}) R^l(\varepsilon^{-\frac{1}{\alpha}-\chi} \mathbf{z}) d\mathbf{z}, \end{aligned} \quad (4.7)$$

where the last equality is followed by changing of variables, the symmetry property $G(t, \mathbf{z}) = G(t, -\mathbf{z})$ of the transition probability density function G , and its semigroup property

$$\int_{\mathbb{R}^n} G(\varepsilon t_{j_1}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_1} - (\mathbf{z} - \mathbf{z}')) G(\varepsilon t_{j_2}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_2} - \mathbf{z}') d\mathbf{z}' = G(\varepsilon(t_{j_1} + t_{j_2}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) - \mathbf{z}).$$

Continue to (4.7), by the spectral representation (2.13) for the k -th power of the covariance function $R(\cdot)$, it is equal to

$$\begin{aligned} &\sum_{j_1, j_2=1}^M a_{j_1} a_{j_2} \varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} G(\varepsilon(t_{j_1} + t_{j_2}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) - \mathbf{z}) \int_{\mathbb{R}^n} e^{i\langle \varepsilon^{-\frac{1}{\alpha}-\chi} \mathbf{z}, \lambda \rangle} f^{*l}(\lambda) d\lambda d\mathbf{z} \\ &= \sum_{j_1, j_2=1}^M a_{j_1} a_{j_2} \varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} e^{i\varepsilon^{-\chi} \langle \lambda, \mathbf{x}_{j_1} - \mathbf{x}_{j_2} \rangle} \widehat{G}(\varepsilon(t_{j_1} + t_{j_2}), \varepsilon^{-\frac{1}{\alpha}-\chi} \lambda) f^{*l}(\lambda) d\lambda \\ &= \sum_{j_1, j_2=1}^M a_{j_1} a_{j_2} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x}_{j_1} - \mathbf{x}_{j_2} \rangle} \exp\{\varepsilon(t_{j_1} + t_{j_2})[\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + |\varepsilon^{-\frac{1}{\alpha}} \lambda|^2)^{\frac{\alpha}{2}}]\} f^{*l}(\varepsilon^{\chi} \lambda) d\lambda \\ &\rightarrow \sum_{j_1, j_2=1}^M a_{j_1} a_{j_2} \sum_{l=N+1}^{\infty} C_l^2 f^{*l}(\mathbf{0}) \int_{\mathbb{R}^n} e^{i\langle \lambda, \mathbf{x}_{j_1} - \mathbf{x}_{j_2} \rangle} \exp\{-(t_{j_1} + t_{j_2})|\lambda|^{\alpha}\} d\lambda < \infty \end{aligned} \quad (4.8)$$

when $\varepsilon \rightarrow 0$, where $f^{*l}(\cdot)$, $l \geq m$, are continuous and uniformly bounded on \mathbb{R}^n since Condition B and Lemma 1 imply:

$$f^{*l}(\lambda) = \int_{\mathbb{R}^n} f^{*m}(\lambda - \eta) f^{*(l-m)}(\eta) d\eta \leq \|B_m\|_{\infty} \int_{\mathbb{R}^n} f^{*(l-m)}(\eta) d\eta = \|B_m\|_{\infty} \quad \forall l > m. \quad (4.9)$$

From (4.8), for any $\delta > 0$ there exists $N_0 \in \mathbb{N}$, $\varepsilon_0 > 0$ such that

$$\mathbb{E}(\xi_{\varepsilon, > N})^2 < \delta, \text{ for any } N \geq N_0, \varepsilon < \varepsilon_0, \quad (4.10)$$

which implies that we suffice to prove a truncated version of (4.4) as follows:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \xi_{\varepsilon, \leq N_0}^p = \begin{cases} 0, & p = 2\nu + 1 \\ (p-1)!! \left\{ \mathbb{E} \left[\left(\sum_{j=1}^M a_j V_{m, N_0}(t_j, \mathbf{x}_j) \right)^2 \right] \right\}^\nu, & p = 2\nu \end{cases}, \quad (4.11)$$

where

$$V_{m, N_0}(t, \mathbf{x}) = \int_{\mathbb{R}^n} e^{i \langle \lambda, \mathbf{x} \rangle} \sigma_{m, N_0} e^{-t|\lambda|^\alpha} W(d\lambda) \text{ with } \sigma_{m, N_0} = \left(\sum_{r=m}^{N_0} f^{*r}(\mathbf{0}) C_r^2 \right)^{\frac{1}{2}}. \quad (4.12)$$

By (4.5) for the definition of $\xi_{\varepsilon, \leq N_0} (= \xi_\varepsilon - \xi_{\varepsilon, > N_0})$, and our rescaling of the initial data, we have

$$\begin{aligned} \mathbb{E}(\xi_{\varepsilon, \leq N_0})^p &= \varepsilon^{-\frac{pn\chi}{2}} \sum_{j_1, \dots, j_p=1}^M \sum_{l_1, \dots, l_p=m}^{N_0} \left[\prod_{i=1}^p a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}} \right] \\ &\quad \times \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_i} - \mathbf{y}_i) \right\} \left[\mathbb{E} \prod_{i=1}^p H_{l_i}(\zeta(\varepsilon^{-\frac{1}{\alpha} - \chi} \mathbf{y}_i)) \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p \\ &= \varepsilon^{-\frac{pn\chi}{2}} \sum_{j_1, \dots, j_p=1}^M \sum_{l_1, \dots, l_p=m}^{N_0} \left[\prod_{i=1}^p a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}} \right] \\ &\quad \times \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_i} - \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left[\mathbb{E} \prod_{i=1}^p H_{l_i}(\zeta(\varepsilon^{-\chi} \mathbf{y}_i)) \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p. \end{aligned} \quad (4.13)$$

To analyze $\mathbb{E}(\xi_{\varepsilon, \leq N_0})^p$, $p = 2\nu$ (the odd $p = 2\nu + 1$ is unnecessary, since all the involved random variables are centered), we employ the diagram method (see, [7] or [12, p.72]). A graph Γ with $l_1 + \cdots + l_p$ vertices is called a (complete) diagram of order (l_1, \dots, l_p) if:

- (a) the set of vertices V of the graph Γ is of the form $V = \bigcup_{j=1}^p W_j$, where $W_j = \{(j, l) : 1 \leq l \leq l_j\}$ is the j -th level of the graph Γ , $1 \leq j \leq p$;
- (b) each vertex is of degree 1; that is, each vertex is just an endpoint of an edge.
- (c) if $((j_1, l_1), (j_2, l_2)) \in \Gamma$ then $j_1 \neq j_2$; that is, the edges of the graph Γ may connect only different levels.

Let $T = T(l_1, \dots, l_p)$ be a set of (complete) diagrams Γ 's of order (l_1, \dots, l_p) . Denote by $E(\Gamma)$ the set of edges of the graph $\Gamma \in T$. For the edge $e = ((j_1, l'_1), (j_2, l'_2)) \in E(\Gamma)$, $j_1 < j_2$, $1 \leq l'_1 \leq l_1$, $1 \leq l'_2 \leq l_2$, we set $d_1(e) = j_1$, $d_2(e) = j_2$, to denote the location of the edge e in Γ . We call a diagram Γ to be *regular* if its levels can be split into pairs in such a manner that no edge connects the levels belonging to different pairs. Denote by $T^* = T^*(l_1, \dots, l_p)$ the set of all regular diagrams in T . Therefore, if $\Gamma \in T^*$ is a regular diagram then it can be divided into $p/2$ sub-diagrams (denoted by $\Gamma_1, \dots, \Gamma_{p/2}$), which can not be separated again; in this case, we naturally define $d_1(\Gamma_i) \equiv d_1(e)$ and $d_2(\Gamma_i) \equiv d_2(e)$ for any $e \in E(\Gamma_i)$, $i = 1, \dots, \nu = p/2$. We denote $\sharp E(\Gamma)$ (resp. $\sharp E(\Gamma_j)$) the number of edges belonging to the specific diagram Γ (resp. the sub-diagram Γ_j).

Based on the notations above and let

$$D_p = \{(J, L) : J = (j_1, \dots, j_p), 1 \leq j_i \leq M, L = (l_1, \dots, l_p), m \leq l_i \leq N_0, i = 1, \dots, p\},$$

(4.13) can be rewritten as

$$\mathbb{E}(\xi_{\varepsilon, \leq N_0})^p = \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T^*} F_\Gamma(J, L, \varepsilon) + \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T \setminus T^*} F_\Gamma(J, L, \varepsilon), \quad (4.14)$$

where

$$K(J, L) = \prod_{i=1}^p a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}} \quad (4.15)$$

$$F_\Gamma(J, L, \varepsilon) = \varepsilon^{-\frac{pn_X}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_i} - \mathbf{y}_i)) \right\} \left[\prod_{e \in E(\Gamma)} R(\varepsilon^{-\chi}(\mathbf{y}_{d_1(e)} - \mathbf{y}_{d_2(e)})) \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p.$$

Next, we want to verify two things:

$$\begin{cases} (1) \lim_{\varepsilon \rightarrow 0} \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T^*} F_\Gamma(J, L, \varepsilon) = (p-1)!! \left\{ \mathbb{E} \left[\left(\sum_{j=1}^M a_j V_{m, N_0}(t_j, \mathbf{x}_j) \right)^2 \right] \right\}^{p/2}, \\ (2) \lim_{\varepsilon \rightarrow 0} \sum_{(J, L) \in D_p} K(J, L) \sum_{\Gamma \in T \setminus T^*} F_\Gamma(J, L, \varepsilon) = 0. \end{cases}$$

Proof of (1): As argued above, each $\Gamma \in T^*$, the the case $p = 2\nu$, $\nu \in \mathbb{N}$, has a unique decomposition into sub-diagrams $\Gamma = (\Gamma_1, \dots, \Gamma_\nu)$, for which each one cannot be further

decomposed. Accordingly, we can rewrite $F_\Gamma(J, L, \varepsilon)$ as the following $\nu = p/2$ products,

$$\begin{aligned}
& F_\Gamma(J, L, \varepsilon) \\
&= \varepsilon^{-\frac{pn\chi}{2}} \prod_{i=1}^{\nu} \int_{\mathbb{R}^{2n}} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{d_1(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{y})) \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{d_2(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_2(\Gamma_i)} - \mathbf{y}')) R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}(\mathbf{y} - \mathbf{y}')) \\
&= \varepsilon^{-\frac{pn\chi}{2}} \prod_{i=1}^{\nu} \int_{\mathbb{R}^n} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} - \mathbf{z})) R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}\mathbf{z}) d\mathbf{z}. \quad (4.16)
\end{aligned}$$

We note that

$$R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}\mathbf{z}) = \varepsilon^{n\chi} \int_{\mathbb{R}^n} e^{i\langle \mathbf{z}, \lambda \rangle} f^{*\sharp E(\Gamma_i)}(\varepsilon^\chi \lambda) d\lambda, \quad i = 1, \dots, \nu, \quad (4.17)$$

and $\sharp E(\Gamma_i) > n/\kappa$ (since $\kappa > n/m$ in the Condition B). By the Fourier transform of G ,

$$\begin{aligned}
& \int_{\mathbb{R}^n} e^{i\langle \mathbf{z}, \lambda \rangle} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} - \mathbf{z})) d\mathbf{z} \\
&= e^{i\langle \lambda, \mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} \rangle} \exp\{\varepsilon(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)})[\mathbf{m} - (\mathbf{m}^{\frac{2}{\alpha}} + |\varepsilon^{-\frac{1}{\alpha}}\lambda|^2)^{\frac{\alpha}{2}}]\}, \quad (4.18)
\end{aligned}$$

applying the the small-scale of G illustrated in (4.2), we have

$$\lim_{\varepsilon \rightarrow 0} F_\Gamma(J, L, \varepsilon) = \prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \int e^{i\langle \lambda, \mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} \rangle} \exp\{-(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)})|\lambda|^\alpha\} d\lambda. \quad (4.19)$$

Meanwhile, if the $L = \{l_1, \dots, l_{2\nu}\}$ in the defining (4.15) of $K(J, L)$ corresponds to a regular diagram Γ in $\mathcal{T}(l_1, \dots, l_{2\nu})$, then

$$K(J, L) = \prod_{i=1}^{\nu} a_{d_1(\Gamma_i)} a_{d_2(\Gamma_i)} \frac{C_{\sharp E(\Gamma_i)}^2}{\sharp E(\Gamma_i)!}. \quad (4.20)$$

Therefore, by (4.19) and (4.20),

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \sum_{(J, L) \in D_{2\nu}} K(J, L) \sum_{\Gamma \in \mathcal{T}^*} F_\Gamma(J, L, \varepsilon) \\
&= \sum_{(J, L) \in D_{2\nu}} \sum_{\Gamma \in \mathcal{T}^*} \left[\prod_{i=1}^{\nu} a_{d_1(\Gamma_i)} a_{d_2(\Gamma_i)} \int e^{i\langle \lambda, \mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} \rangle} \exp\{-(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)})|\lambda|^\alpha\} d\lambda \right] \\
&\quad \times \left[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \frac{C_{\sharp E(\Gamma_i)}^2}{\sharp E(\Gamma_i)!} \right]. \quad (4.21)
\end{aligned}$$

We note that all components in the first bracket in (4.21) are independent to the index set L and the summation $\sum_{\Gamma \in T^*}$ depends only on \sum_L , therefore

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \sum_{(J,L) \in D_{2\nu}} K(J,L) \sum_{\Gamma \in T^*} F_\Gamma(J,L,\varepsilon) \\
&= \sum_L \sum_{\Gamma \in T^*} \sum_J \left[\prod_{i=1}^\nu a_{d_1(\Gamma_i)} a_{d_2(\Gamma_i)} \int e^{i\langle \lambda, \mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} \rangle} \exp\{-(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)})|\lambda|^\alpha\} d\lambda \right] \\
&\quad \times \left[\prod_{i=1}^\nu f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \frac{C_{\sharp E(\Gamma_i)}^2}{\sharp E(\Gamma_i)!} \right] \\
&= \left[\sum_{j,j'=1}^M a_j a_{j'} \int e^{i\langle \lambda, \mathbf{x}_j - \mathbf{x}_{j'} \rangle} \exp\{-(t_j + t_{j'})|\lambda|^\alpha\} d\lambda \right]^\nu \sum_L \sum_{\Gamma \in T^*} \left[\prod_{i=1}^\nu f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \frac{C_{\sharp E(\Gamma_i)}^2}{\sharp E(\Gamma_i)!} \right].
\end{aligned} \tag{4.22}$$

To handle the summation in the above, we note that $\prod_{i=1}^\nu f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \frac{C_{\sharp E(\Gamma_i)}^2}{\sharp E(\Gamma_i)!}$ only depends on $\{\sharp E(\Gamma_i), i = 1, \dots, \nu\}$, not on the structures of sub-diagrams Γ_i , $i = 1, \dots, \nu$; thus we may rewrite the above summation based on the following observation. Let s be the number of different integers r_1, \dots, r_s in $\{l_1, \dots, l_{2\nu}\}$ with $m \leq r_1 < \dots < r_s \leq N_0$. A regular diagram requires $1 \leq s \leq \nu$, which also implies that the set $\{l_1, \dots, l_{2\nu}\}$ can be split into s subsets Q_1, \dots, Q_s and all elements within Q_i have the common value r_i , $i = 1, \dots, s$. For the number of *pairs* within each subset Q_i , we denote it by q_i , which satisfies $q_i \geq 1$, $i = 1, \dots, s$, and $q_1 + \dots + q_s = \nu$. Thus, the above summation is

$$\sum_{1 \leq s \leq \nu} (s!) \sum_{m \leq r_1 < \dots < r_s = N_0} \sum_{q_1 + \dots + q_s = \nu} \frac{(2\nu)!}{(2q_1)! \dots (2q_s)!} [\dots].$$

However, for any $(s; r_1, \dots, r_s; q_1, \dots, q_s)$ in the above sum, it corresponds $\frac{(2q_1)! \dots (2q_s)!}{2^\nu q_1! \dots q_s!} (r_1!)^{q_1} \dots (r_s!)^{q_s}$

different regular diagrams. Therefore,

$$\begin{aligned}
& \sum_L \sum_{\Gamma \in \mathbf{T}^*} \left[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \frac{C_{\sharp E(\Gamma_i)}^2}{\sharp E(\Gamma_i)!} \right] \\
&= \sum_{1 \leq s \leq \nu} (s!) \sum_{m \leq r_1 < \dots < r_s = N_0} \sum_{q_1 + \dots + q_s = \nu} \frac{(2\nu)!}{2^\nu q_1! \dots q_s!} (r_1!)^{q_1} \dots (r_s!)^{q_s} \left[\prod_{i=1}^s (f^{*r_i}(\mathbf{0}) \frac{C_{r_i}^2}{r_i!})^{q_i} \right] \\
&= (2\nu - 1)!! \sum_{1 \leq s \leq \nu} (s!) \sum_{m \leq r_1 < \dots < r_s = N_0} \sum_{q_1 + \dots + q_s = \nu} \frac{\nu!}{q_1! \dots q_s!} \left[\prod_{i=1}^s (f^{*r_i}(\mathbf{0}) C_{r_i}^2)^{q_i} \right] \\
&= (2\nu - 1)!! \left[\sum_{r=m}^{N_0} f^{*r}(\mathbf{0}) C_r^2 \right]^\nu. \tag{4.23}
\end{aligned}$$

Substituting (4.23) into (4.22) and recalling $\sigma_{m,N_0} = (\sum_{r=m}^{N_0} f^{*r}(\mathbf{0}) C_r^2)^{\frac{1}{2}}$, we get

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \sum_{(J,L) \in D_{2\nu}} K(J, L) \sum_{\Gamma \in \mathbf{T}^*} F_\Gamma(J, L, \varepsilon) \\
&= (2\nu - 1)!! \left[\sum_{j,j'=1}^M a_j a_{j'} \int_{\mathbb{R}^n} e^{i \langle \lambda, \mathbf{x}_j - \mathbf{x}_{j'} \rangle} \exp\{-(t_j + t_{j'})|\lambda|^\alpha\} d\lambda \right]^\nu \left[\sum_{r=m}^{N_0} f^{*r}(\mathbf{0}) C_r^2 \right]^\nu \\
&= (2\nu - 1)!! \left[\mathbb{E} \left(\sum_{j=1}^M a_j \int_{\mathbb{R}^n} e^{i \langle \lambda, \mathbf{x}_j \rangle} \sigma_{m,N_0} e^{-t_j |\lambda|^\alpha} W(d\lambda) \right)^2 \right]^\nu. \tag{4.24}
\end{aligned}$$

Proof of (2): $\lim_{\varepsilon \rightarrow 0} \sum_{(J,L) \in D_p} K(J, L) \sum_{\Gamma \in \mathbf{T} \setminus \mathbf{T}^*} F_\Gamma(J, L, \varepsilon) = 0$.

By (4.11), the number of elements in the summation of $\sum_{(J,L) \in D_p}$ is finite, thus it suffices to show that $\lim_{\varepsilon \rightarrow 0} F_\Gamma(J, L, \varepsilon) = 0$, i.e.,

$$\varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_i} - \mathbf{y}_i)) \right\} \left[\prod_{e \in E(\Gamma)} R(\varepsilon^{-\chi}(\mathbf{y}_{d_1(e)} - \mathbf{y}_{d_2(e)})) \right] d\mathbf{y}_1 \dots d\mathbf{y}_p \rightarrow 0 \tag{4.25}$$

for each $\Gamma \in \mathbf{T}(l_1, \dots, l_p) \setminus \mathbf{T}^*$. With loss of generality, we may just prove (4.25) for $t_{j_i} = 1$ and $\mathbf{x}_{j_i} = \mathbf{0}$, $i = 1, \dots, p$, and also just consider the case $l_1 \leq l_2 \leq \dots \leq l_p$. Let

$$A_{j,j'} := \{e \in E(\Gamma) \mid d_1(e) = j, d_2(e) = j'\}, \quad B(i) := \bigcup_{j' > i} A_{i,j'}, \quad 1 \leq i, j < j' \leq p.$$

We observe that the number $\sharp B(i)$ of $B(i)$ must be $\leq l_i$, and a non-regular diagram Γ must contain and a non-empty $B(i)$ with $\sharp B(i) < l_i$; moreover, it has ([7, (2.20)])

$$\sum_{i=1}^p \frac{\sharp B(i)}{l_i} \geq \frac{p}{2}. \tag{4.26}$$

$$\begin{aligned}
& F_{\Gamma}(J, L, \varepsilon) \\
&= \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left[\prod_{i; B(i) \neq \phi} \prod_{e \in B(i)} R(\varepsilon^{-\chi}(\mathbf{y}_i - \mathbf{y}_{d_2(e)})) \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p \\
&\leq \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left[\prod_{i; B(i) \neq \phi} \sum_{e \in B(i)} \frac{1}{\#B(i)} R^{\#B(i)}(\varepsilon^{-\chi}(\mathbf{y}_i - \mathbf{y}_{d_2(e)})) \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p \\
&\leq c \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left[\prod_{i; B(i) \neq \phi} \sum_{j; A_{i,j} \neq \phi} \frac{1}{\#B(i)} R^{\#B(i)}(\varepsilon^{-\chi}(\mathbf{y}_i - \mathbf{y}_j)) \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p,
\end{aligned} \tag{4.27}$$

where $c = \prod_{i; B(i) \neq \phi} \sum_{j; A_{i,j} \neq \phi} \#A_{i,j} / \#B(i)$.

For any $i \in \{1, \dots, p-1\}$ with $B(i) \neq \phi$, let $j(i)$ be any term in $\{j'; A_{i,j'} \neq \phi\}$. To prove (4.27) $\rightarrow 0$, by the spectral representation, it suffices to show that

$$\varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left[\prod_{i; B(i) \neq \phi} \int e^{i \langle \mathbf{y}_i - \mathbf{y}_{j(i)}, \lambda_{i,j(i)} \rangle} f^{*\#B(i)}(\varepsilon^{\chi} \lambda_{i,j(i)}) \varepsilon^{n\chi} d\lambda_{i,j(i)} \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p \tag{4.28}$$

converges to zero when $\varepsilon \rightarrow 0$.

Applying Lemma 1 to $k = \#B(i)$, the number of $B(i)$, we see that,

$$f^{*\#B(i)}(\lambda) \leq \begin{cases} o(1), & \text{if } \#B(i) = l_i, \\ o(|\lambda|^{n(\frac{\#B(i)}{l_i}-1)}), & \text{if } 1 < \#B(i) < l_i, \end{cases} \quad \text{when } |\lambda| \rightarrow 0. \tag{4.29}$$

For example, in the case (a) of Lemma 1, we can write it as follows:

$$f^{*\#B(i)}(\lambda) = C_{\#B(i)}(\lambda) |\lambda|^{\#B(i) \frac{n}{l_i} - n}, \quad C_{\#B(i)}(\lambda) = B_{\#B(i)}(\lambda) |\lambda|^{\#B(i)(\kappa - \frac{n}{l_i})}, \tag{4.30}$$

where $\lim_{|\lambda| \rightarrow 0} C_{\#B(i)}(\lambda) = 0$ because $\kappa > n/m \geq n/l_i$.

Thus,

$$(4.28) \leq \varepsilon^{-\frac{pn\chi}{2}} o(\varepsilon^{\chi n (\sum \frac{\#B(i)}{l_i})}) Q_{\varepsilon}, \tag{4.31}$$

where

$$Q_{\varepsilon} = \int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left[\prod_{i; B(i) \neq \phi} \int e^{i \langle \mathbf{y}_i - \mathbf{y}_{j(i)}, \lambda_{i,j(i)} \rangle} |\lambda_{i,j(i)}|^{n(\frac{\#B(i)}{l_i}-1)} d\lambda_{i,j(i)} \right] d\mathbf{y}_1 \cdots d\mathbf{y}_p,$$

which is bounded in $0 < \epsilon \ll 1$. Because, firstly, for each $\lambda_{i,j(i)}$, by (4.2) the following is bounded in $0 < \epsilon \ll 1$,

$$\int_{\mathbb{R}^{np}} \left\{ \prod_{i=1}^p \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \right\} \left\{ \prod_{i; B(i) \neq \phi} e^{i \langle \mathbf{y}_i - \mathbf{y}_{j(i)}, \lambda_{i,j(i)} \rangle} \right\} d\mathbf{y}_1 \cdots d\mathbf{y}_p, \quad (4.32)$$

and moreover

$$\prod_{i; B(i) \neq \phi} |\lambda_{i,j(i)}|^{n(\frac{\#B(i)}{l_i} - 1)}$$

is integrable with respect to $\prod_{i; B(i) \neq \phi} d\lambda_{i,j(i)}$ near the origin. Finally, the convergence of (4.31) to zero is followed by the inequality cited above, i.e. $\sum_{i=1}^p \frac{\#B(i)}{l_i} \geq \frac{p}{2}$. \square

Proof of Theorem 3.

By the solution form (2.2) and $\int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(t, \mathbf{x}) d\mathbf{x} = 1$,

$$\begin{aligned} & T^{\frac{m\kappa}{4}} \left\{ u(Tt, \sqrt{T}\mathbf{x}; h(\zeta(\cdot))) - C_0 \right\} \\ &= T^{\frac{m\kappa}{4}} \left\{ \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}) \left[C_0 + \sum_{k=m}^{\infty} C_k \frac{H_k(\zeta(\mathbf{y}))}{\sqrt{k!}} \right] d\mathbf{y} - C_0 \right\} \\ &= \sum_{k=m}^{\infty} T^{\frac{m\kappa}{4}} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}) H_k(\zeta(\mathbf{y})) d\mathbf{y} =: \sum_{k=m}^{\infty} u_{k,T}(t, \mathbf{x}). \end{aligned} \quad (4.33)$$

By the Slutsky argument (see, for example, [18, p. 6.]), Theorem 3 will be proved if we can show that

$$\begin{cases} \text{(i)} & u_{m,T}(t, \mathbf{x}) \Rightarrow U_m(t, \mathbf{x}), \\ \text{(ii)} & \sum_{k=m+1}^{\infty} u_{k,T}(t, \mathbf{x}) \rightarrow 0 \text{ in probability,} \end{cases} \quad \text{as } T \rightarrow \infty. \quad (4.34)$$

Proof of (i): Replacing the component $H_m(\zeta(\mathbf{y}))$ in the expression of $u_{m,T}(t, \mathbf{x})$ with its Itô-Wiener expansion (2.10), and using the Fourier transform $\widehat{G}_{\alpha, \mathbf{m}}(t, \lambda)$ of $G_{\alpha, \mathbf{m}}(t, \mathbf{x})$ in (2.1), we have

$$\begin{aligned} & u_{m,T}(t, \mathbf{x}) \\ &= T^{\frac{m\kappa}{4}} \frac{C_m}{\sqrt{m!}} \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}) \left\{ \int'_{\mathbb{R}^{n \times m}} e^{i \langle \mathbf{y}, \lambda_1 + \dots + \lambda_m \rangle} \prod_{\sigma=1}^m \sqrt{f(\lambda_{\sigma})} W(d\lambda_{\sigma}) \right\} d\mathbf{y} \\ &= T^{\frac{m\kappa}{4}} \frac{C_m}{\sqrt{m!}} \int'_{\mathbb{R}^{n \times m}} e^{i \langle \sqrt{T}\mathbf{x}, \lambda_1 + \dots + \lambda_m \rangle} \widehat{G}_{\alpha, \mathbf{m}}(Tt, \lambda_1 + \dots + \lambda_m) \prod_{\sigma=1}^m \sqrt{f(\lambda_{\sigma})} W(d\lambda_{\sigma}). \end{aligned} \quad (4.35)$$

By the definition about $\int'_{\mathbb{R}^{n \times m}}$ in (2.10) and the self-similarity property $W(T^{-\frac{1}{2}}d\lambda) \stackrel{d}{=} T^{-\frac{n}{4}}W(d\lambda)$, $u_{m,T}$ has the same finite dimensional distributions ($\stackrel{d}{=}$) as $\tilde{u}_{m,T}$, where

$$\begin{aligned} \tilde{u}_{m,T}(t, \mathbf{x}) &= \frac{C_m}{\sqrt{m!}} T^{\frac{m(\kappa-n)}{4}} \int'_{\mathbb{R}^{n \times m}} e^{i\langle \mathbf{x}, \lambda_1 + \dots + \lambda_m \rangle} \widehat{G}_{\alpha, \mathbf{m}}(Tt, T^{-\frac{1}{2}}(\lambda_1 + \dots + \lambda_m)) \\ &\quad \times \prod_{\sigma=1}^m \sqrt{f(T^{-\frac{1}{2}}\lambda_\sigma)} W(d\lambda_\sigma). \end{aligned} \quad (4.36)$$

From the isometry property of the multiple Wiener integrals and the integral representation of the limiting field $U_m(t, \mathbf{x})$ in (3.5),

$$\begin{aligned} &\mathbb{E}|\tilde{u}_{m,T}(t, \mathbf{x}) - U_m(t, \mathbf{x})|^2 \\ &= C_m^2 \int_{\mathbb{R}^{nm}} \left| T^{\frac{m(\kappa-n)}{4}} \widehat{G}_{\alpha, \mathbf{m}}(Tt, T^{-\frac{1}{2}}(\lambda_1 + \dots + \lambda_m)) \prod_{\sigma=1}^m \sqrt{f(T^{-\frac{1}{2}}\lambda_\sigma)} \right. \\ &\quad \left. - B(\mathbf{0})^{\frac{m}{2}} \frac{\exp(-t\frac{\alpha}{2}\mathbf{m}^{1-\frac{2}{\alpha}}|\lambda_1 + \dots + \lambda_m|^2)}{(|\lambda_1| \dots |\lambda_m|)^{\frac{n-\kappa}{2}}} \right|^2 \prod_{\sigma=1}^m d\lambda_\sigma. \end{aligned} \quad (4.37)$$

Condition C and (4.1) allow us to apply the dominated convergence theorem to show that (4.37) will converge to zero when $T \rightarrow \infty$. We note that the convergence in (4.1) can be shown to be monotone decreasing when $T \uparrow \infty$ for each $t > 0$ and $\lambda \in \mathbb{R}^n$.

Thus, we get

$$\lim_{T \rightarrow \infty} \mathbb{E}|\tilde{u}_{m,T}(t, \mathbf{x}) - U_m(t, \mathbf{x})|^2 = 0, \quad (4.38)$$

and the claim (i) is concluded by $u_{m,T} \stackrel{d}{=} \tilde{u}_{m,T}$ and the Cramer-Wold theorem.

Proof of (ii): By the orthogonal property (2.9), the semigroup property of $G_{\alpha, \mathbf{m}}(t, \mathbf{x})$, and (2.13), we have the following equalities

$$\begin{aligned}
& \mathbb{E}\left[\left(\sum_{k=m+1}^{\infty} u_{k,T}(t, \mathbf{x})\right)^2\right] \\
&= T^{\frac{m\kappa}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}) G_{\alpha, \mathbf{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}') R^k(\mathbf{y} - \mathbf{y}') d\mathbf{y} d\mathbf{y}' \\
&= T^{\frac{m\kappa}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} G_{\alpha, \mathbf{m}}(2Tt, \mathbf{z}) R^k(\mathbf{z}) d\mathbf{z} \\
&= T^{\frac{m\kappa}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha, \mathbf{m}}(2Tt, \lambda) f^{*k}(\lambda) d\lambda \quad (\text{by (2.13)}) \\
&= T^{\frac{m\kappa-n}{2}} \left(\sum_{k=m+1}^{k^*} + \sum_{k=k^*+1}^{\infty} \right) C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha, \mathbf{m}}(2Tt, T^{-\frac{1}{2}}\lambda) f^{*k}(T^{-\frac{1}{2}}\lambda) d\lambda =: (I) + (II), \quad (4.39)
\end{aligned}$$

where $k^* = \max\{k \in \mathbb{N} \mid k \geq m+1, k\kappa \leq n\}$.

For the case $k^*\kappa < n$, by Lemma 1 and (4.1),

$$\begin{aligned}
\lim_{T \rightarrow \infty} (I) &= \lim_{T \rightarrow \infty} T^{\frac{m\kappa-n}{2}} \sum_{k=m+1}^{k^*} C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha, \mathbf{m}}(2Tt, T^{-\frac{1}{2}}\lambda) B_k(T^{-\frac{1}{2}}\lambda) |T^{-\frac{1}{2}}\lambda|^{k\kappa-n} d\lambda \\
&\leq \lim_{T \rightarrow \infty} \sum_{k=m+1}^{k^*} T^{\frac{m\kappa-k\kappa}{2}} C_k^2 \|B_k\|_{\infty} \int_{\mathbb{R}^n} e^{-t\frac{\alpha}{2}\mathbf{m}^1 - \frac{2}{\alpha}|\lambda|^2} |\lambda|^{k\kappa-n} d\lambda \\
&\leq \lim_{T \rightarrow \infty} T^{-\frac{\kappa}{2}} \sum_{k=m+1}^{k^*} C_k^2 \|B_k\|_{\infty} \int_{\mathbb{R}^n} e^{-t\frac{\alpha}{2}\mathbf{m}^1 - \frac{2}{\alpha}|\lambda|^2} |\lambda|^{k\kappa-n} d\lambda = 0.
\end{aligned}$$

For the case $k^*\kappa = n$, we still have $\lim_{T \rightarrow \infty} (I) = 0$ because

$$\lim_{T \rightarrow \infty} T^{\frac{m\kappa-n}{2}} C_{k^*}^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha, \mathbf{m}}(2Tt, T^{-\frac{1}{2}}\lambda) B_{k^*}(T^{-\frac{1}{2}}\lambda) \ln(2 + T^{\frac{1}{2}}|\lambda|^{-1}) d\lambda = 0.$$

On the other hand, by the assumption $m\kappa < n$ in Condition C and Lemma 1, for any $k > k^* + 1$ we have $\|f^{*k}\|_{\infty} \leq \|f^{*(k^*+1)}\|_{\infty}$, so

$$\lim_{T \rightarrow \infty} (II) \leq \lim_{T \rightarrow \infty} T^{\frac{m\kappa-n}{2}} \sum_{k=k^*+1}^{\infty} C_k^2 \|f^{*(k^*+1)}\|_{\infty} \int_{\mathbb{R}^n} \widehat{G}_{\alpha, \mathbf{m}}(2Tt, T^{-\frac{1}{2}}\lambda) d\lambda = 0.$$

Hence $\lim_{T \rightarrow \infty} \mathbb{E}\left[\left(\sum_{k=m+1}^{\infty} u_{k,T}(t, \mathbf{x})\right)^2\right] = 0$ and the claim (ii) is proved by the Markov inequality. \square

Proof of Theorem 4.

The following proof is a hybrid of the proofs of Theorems 2 and 3, we give a full presentation mainly to see how the rescaling of the initial data is proceeded. By the Hermite expansion and the solution form (2.2) we can rewrite

$$u^\varepsilon(t, \mathbf{x}) = \sum_{k=m}^{\infty} \varepsilon^{-\frac{\chi m \kappa}{2}} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G_{\alpha, m}(\varepsilon t, \mathbf{y}) H_k(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}} \mathbf{x} - \mathbf{y}))) d\mathbf{y} =: \sum_{k=m}^{\infty} I_k^\varepsilon(t, \mathbf{x}). \quad (4.40)$$

By the Slutsky argument again, we show that

$$\begin{cases} \text{(i)} & I_m^\varepsilon(t, \mathbf{x}) \Rightarrow V_m(t, \mathbf{x}), \\ \text{(ii)} & \sum_{k=m+1}^{\infty} I_k^\varepsilon(t, \mathbf{x}) \rightarrow 0 \text{ in probability,} \end{cases} \quad \text{as } \varepsilon \rightarrow 0. \quad (4.41)$$

Proof of (i): By substituting the Itô-Wiener expansion (2.10) for the random field $H_m(\zeta(\cdot))$ into $I_m^\varepsilon(t, \mathbf{x})$ and exchanging the order of integration

$$\begin{aligned} & I_m^\varepsilon(t, \mathbf{x}) \\ &= \frac{C_m}{\sqrt{m!}} \varepsilon^{-\frac{\chi m \kappa}{2}} \int_{\mathbb{R}^n} G_{\alpha, m}(\varepsilon t, \mathbf{y}) H_m(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}} \mathbf{x} - \mathbf{y}))) d\mathbf{y} \\ &= \frac{C_m}{\sqrt{m!}} \varepsilon^{-\frac{\chi m \kappa}{2}} \int_{\mathbb{R}^n} G_{\alpha, m}(\varepsilon t, \mathbf{y}) \int_{\mathbb{R}^{n \times m}}' e^{i \langle \varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}} \mathbf{x} - \mathbf{y}), \lambda_1 + \dots + \lambda_m \rangle} \prod_{\sigma=1}^m \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) d\mathbf{y} \\ &= \frac{C_m}{\sqrt{m!}} \varepsilon^{-\frac{\chi m \kappa}{2}} \int_{\mathbb{R}^{n \times m}}' e^{i \langle \varepsilon^{-\chi} \mathbf{x}, \lambda_1 + \dots + \lambda_m \rangle} \widehat{G}_{\alpha, m}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}-\chi}(\lambda_1 + \dots + \lambda_m)) \prod_{\sigma=1}^m \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) d\mathbf{y} \\ &\stackrel{d}{=} \frac{C_m}{\sqrt{m!}} \varepsilon^{\frac{\chi m(n-\kappa)}{2}} \int_{\mathbb{R}^{n \times m}}' e^{i \langle \mathbf{x}, \lambda'_1 + \dots + \lambda'_m \rangle} \widehat{G}_{\alpha, m}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}}(\lambda'_1 + \dots + \lambda'_m)) \prod_{\sigma=1}^m \sqrt{f(\varepsilon^\chi \lambda'_\sigma)} W(d\lambda'_\sigma) \\ &=: \widetilde{I}_m^\varepsilon(t, \mathbf{x}), \end{aligned} \quad (4.42)$$

where we have used the self-similarity property $W(\varepsilon^\chi d\lambda) \stackrel{d}{=} \varepsilon^{\frac{n\chi}{2}} W(d\lambda)$ in the last equality.

Now, applying the isometry property of the multiple Wiener integrals to the difference

of $\tilde{I}_m^\varepsilon(t, \mathbf{x})$ and the random field $V_m(t, \mathbf{x})$ in (3.7), we have

$$\begin{aligned} \mathbb{E}|\tilde{I}_m^\varepsilon(t, \mathbf{x}) - V_m(t, \mathbf{x})|^2 &= C_m^2 \int_{\mathbb{R}^{nm}} \left| \varepsilon^{\frac{\chi m(n-\kappa)}{2}} \hat{G}_{\alpha, m}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}}(\lambda_1 + \dots + \lambda_m)) \prod_{\sigma=1}^m \sqrt{f(\varepsilon^\chi \lambda_\sigma)} \right. \\ &\quad \left. - B(\mathbf{0})^{\frac{m}{2}} e^{-t|\lambda_1 + \dots + \lambda_m|^\alpha} (|\lambda_1| \dots |\lambda_m|)^{\frac{\kappa-n}{2}} \right|^2 \prod_{\sigma=1}^m d\lambda_\sigma \rightarrow 0 \end{aligned} \quad (4.43)$$

when $\varepsilon \rightarrow 0$, by Condition C and (4.2).

By the Markov inequality, (4.43) implies $\tilde{I}_m^\varepsilon(t, \mathbf{x}) \rightarrow V_m(t, \mathbf{x})$ in probability. However, because $I_m^\varepsilon(t, \mathbf{x}) \stackrel{d}{=} \tilde{I}_m^\varepsilon(t, \mathbf{x})$, the claim (i) is concluded, by the Cramer-Wold argument.

Proof of (ii): From (4.40), by the orthogonal property (2.9) and the semigroup property of $G_{\alpha, m}(t, \mathbf{x})$,

$$\begin{aligned} \mathbb{E} \left(\sum_{k=m+1}^{\infty} I_k^\varepsilon(t, \mathbf{x}) \right)^2 &= \sum_{k=m+1}^{\infty} \mathbb{E} (I_k^\varepsilon(t, \mathbf{x}))^2 \\ &= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi m \kappa} C_k^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_{\alpha, m}(\varepsilon t, \mathbf{y}) G_{\alpha, m}(\varepsilon t, \mathbf{y}') R^k(\varepsilon^{-\frac{1}{\alpha} - \chi}(\mathbf{y} - \mathbf{y}')) d\mathbf{y} d\mathbf{y}' \\ &= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi m \kappa} C_\rho^2 \int_{\mathbb{R}^n} G_{\alpha, m}(2\varepsilon t, \mathbf{z}) R^k(\varepsilon^{-\frac{1}{\alpha} - \chi} \mathbf{z}) d\mathbf{z} \\ &= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi m \kappa} C_k^2 \int_{\mathbb{R}^n} \hat{G}_{\alpha, m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha} - \chi} \lambda) f^{*k}(\lambda) d\lambda \\ &= \left(\sum_{k=m+1}^{k^*} + \sum_{k=k^*+1}^{\infty} \right) \varepsilon^{\chi(n-m\kappa)} C_k^2 \int_{\mathbb{R}^n} \hat{G}_{\alpha, m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} \lambda) f^{*k}(\varepsilon^\chi \lambda) d\lambda =: (I) + (II), \end{aligned}$$

where $k^* = \max\{k \in \mathbb{N} \mid k \geq m+1, k\kappa \leq n\}$.

For the case $k^*\kappa < n$, by Lemma 1,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} (I) &= \lim_{\varepsilon \rightarrow 0} \sum_{k=m+1}^{k^*} \varepsilon^{\chi(n-m\kappa)} C_k^2 \int_{\mathbb{R}^n} \hat{G}_{\alpha, m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} \lambda) B_k(\varepsilon^\chi \lambda) |\varepsilon^\chi \lambda|^{k\kappa-n} d\lambda \\ &\leq \lim_{\varepsilon \rightarrow 0} \sum_{k=m+1}^{k^*} \varepsilon^{\chi\kappa(k-m)} C_k^2 \|B_k\|_\infty \int_{\mathbb{R}^n} e^{-2t|\lambda|^\alpha} |\lambda|^{k\kappa-n} d\lambda = 0. \end{aligned}$$

For the case, $k^*\kappa = n$, we still have $\lim_{\varepsilon \rightarrow 0} (I) = 0$ because

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\chi(n-m\kappa)} C_{k^*}^2 \int_{\mathbb{R}^n} \hat{G}_{\alpha, m}(2\varepsilon t, \varepsilon^{-\frac{1}{\alpha}} \lambda) B_{k^*}(\varepsilon^\chi \lambda) \ln(2 + |\varepsilon^\chi \lambda|^{-1}) d\lambda = 0.$$

On the other hand, by the assumption $\kappa < n/m$ in Condition C and Lemma 1, for any $k > k^* + 1$ we have $\|f^{*k}\|_\infty \leq \|f^{*(k^*+1)}\|_\infty$, so

$$\lim_{\varepsilon \rightarrow 0} (II) \leq \lim_{\varepsilon \rightarrow 0} \sum_{k=k^*+1}^{\infty} \varepsilon^{\chi(n-m\kappa)} C_k^2 \|f^{*(k^*+1)}\|_\infty \int_{\mathbb{R}^n} e^{-2t|\lambda|^\alpha} d\lambda = 0.$$

Hence $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[(\sum_{k=m+1}^{\infty} I_k^\varepsilon(t, \mathbf{x}))^2] = 0$ and the claim (ii) is proved by the Markov inequality. \square

Appendix: Proof of Lemma 1.

The idea of following proofs comes from [28, p.115, Theorem 3] and [16, p.160, Theorem 8.8]. We only consider their results for the density functions on the whole space. Suppose that two spectral density functions f_1 and f_2 are in the form

$$0 \leq f_j(\lambda) = \frac{K_j(\lambda)}{|\lambda|^{n-\kappa_j}}, \quad \kappa_j > 0, \quad j = 1, 2, \quad (4.44)$$

where $K_1(\lambda)$ and $K_2(\lambda)$ are nonnegative functions belonging to $C(\mathbb{R}^n \setminus \{\mathbf{0}\})$.

Let $g(\lambda) = \int_{\mathbb{R}^n} f_1(\lambda - \eta) f_2(\eta) d\eta$, $\lambda \in \mathbb{R}^n$. To prove Lemma 1, we show that g can be written as

$$g(\lambda) = \begin{cases} B(\lambda) |\lambda|^{\kappa_1 + \kappa_2 - n}, & \text{for } \kappa_1 + \kappa_2 < n, \\ B(\lambda) \ln(2 + \frac{1}{|\lambda|}), & \text{for } \kappa_1 + \kappa_2 = n, \\ B(\lambda) \in C(\mathbb{R}^n), & \text{for } \kappa_1 + \kappa_2 > n, \end{cases}$$

for some bounded function $B(\lambda) \in C(\mathbb{R}^n \setminus \{\mathbf{0}\})$.

Case 1: $\kappa_1 + \kappa_2 < n$. For any $\lambda_0 \neq \mathbf{0}$, we divide \mathbb{R}^n into four parts: $\mathbb{R}^n = D_1 \cup D_2 \cup D_3 \cup D_4$, where

$$\begin{aligned} D_1 &= \{\eta \in \mathbb{R}^n \mid |\eta - \lambda_0| < |\lambda_0|/2\}, \\ D_2 &= \{\eta \in \mathbb{R}^n \mid |\eta| < |\lambda_0|/2\}, \\ D_3 &= \{\eta \in (D_1 \cup D_2)^c \mid |\eta - \lambda_0| < |\eta|\}, \\ D_4 &= \{\eta \in (D_1 \cup D_2)^c \mid |\eta - \lambda_0| > |\eta|\}. \end{aligned}$$

Therefore,

$$g(\lambda_0) = \sum_{j=1}^4 \int_{D_j} f_1(\lambda_0 - \eta) f_2(\eta) d\eta =: I_1 + I_2 + I_3 + I_4.$$

$$\begin{aligned}
I_1(\lambda_0) &\leq \left(\sup_{\eta' \in D_1} f_2(\eta') \right) \int_{D_j} f_1(\lambda_0 - \eta) d\eta \\
&\leq \left(\sup_{\eta \in D_1} K_2(\eta) \right) \left(\frac{|\lambda_0|}{2} \right)^{\kappa_2 - n} \left(\sup_{\eta \in D_2} K_1(\eta) \right) c_n \int_0^{\frac{|\lambda_0|}{2}} r^{\kappa_1 - 1} dr = C |\lambda_0|^{\kappa_1 + \kappa_2 - n}
\end{aligned}$$

where c_n is the surface area of the unit sphere on \mathbb{R}^n and C is a constant independent to λ_0 . Similarly, $I_2(\lambda_0) \leq C |\lambda_0|^{\kappa_1 + \kappa_2 - n}$.

By the fact $I_3 \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}) \cap L^1(\mathbb{R}^n)$, we know $\sup_{|\lambda_0| \geq 1} I_3(\lambda_0) < \infty$. So we suffice to study the behavior of $I_3(\cdot)$ on the domain $\{\lambda_0 \mid |\lambda_0| < 1\}$.

By the requirement (4.44), $\lim_{|\eta| \rightarrow \infty} K_j(\eta) |\eta|^{\kappa_j} = 0$; that is, for any $\varepsilon > 0$, there exists a constant $M = M(\varepsilon) > 0$ such that

$$K_j(\eta) \leq \varepsilon |\eta|^{-\kappa_j} \quad \text{for all } |\eta| > M. \quad (4.45)$$

Because $|\eta - \lambda_0| < |\eta|$ for $\eta \in D_3$,

$$I_3(\lambda_0) \leq \left(\int_{D_3 \cap \{|\eta - \lambda_0| > M+1\}} + \int_{D_3 \cap \{|\eta - \lambda_0| < M+1\}} \right) \frac{K_1(\lambda_0 - \eta) K_2(\eta)}{|\lambda_0 - \eta|^{2n - \kappa_1 - \kappa_2}} d\eta =: I_{3,1}(\lambda_0) + I_{3,2}(\lambda_0).$$

By using (4.45) and $|\eta - \lambda_0| < |\eta|$ again,

$$\begin{aligned}
I_{3,1}(\lambda_0) &\leq \varepsilon \int_{D_3 \cap \{|\eta - \lambda_0| > M+1\}} \frac{K_1(\lambda_0 - \eta) |\eta|^{-\kappa_2}}{|\lambda_0 - \eta|^{2n - \kappa_1 - \kappa_2}} d\eta \leq \varepsilon \int_{\{|\eta - \lambda_0| > M+1\}} \frac{K_1(\lambda_0 - \eta)}{|\lambda_0 - \eta|^{2n - \kappa_1}} d\eta \\
&\leq \varepsilon (M+1)^{-n} \int_{\mathbb{R}^n} \frac{K_1(\eta)}{|\eta|^{n - \kappa_1}} d\eta = \varepsilon (M+1)^{-n}
\end{aligned} \quad (4.46)$$

$$I_{3,2}(\lambda_0) \leq \|K_1\|_\infty \|K_2\|_\infty c_n \int_{\frac{\lambda_0}{2}}^{M+1} \frac{r^{n-1}}{r^{2n - \kappa_1 - \kappa_2}} dr < \frac{\|K_1\|_\infty \|K_2\|_\infty c_n}{n - \kappa_1 - \kappa_2} \left(\frac{|\lambda_0|}{2} \right)^{\kappa_1 + \kappa_2 - n}. \quad (4.47)$$

Combining (4.46) and (4.47), we get $I_3(\lambda) = B(\lambda) |\lambda|^{\kappa_1 + \kappa_2 - n}$ for some bounded function B . This observation still holds for I_4 . Therefore, the proof for the case $\kappa_1 + \kappa_2 < n$ is finished.

Case 2: $\kappa_1 + \kappa_2 = n$, $\kappa_1, \kappa_2 > 0$. Let $\hat{\lambda}_0 = \lambda_0 / |\lambda_0|$,

$$\begin{aligned}
g(\lambda_0) &= \int_{\mathbb{R}^n} \frac{K_1(\lambda_0 - \eta) K_2(\eta)}{|\lambda_0 - \eta|^{n - \kappa_1} |\eta|^{n - \kappa_2}} d\eta = \left(\int_{\{|\eta| < 2\}} + \int_{\{|\eta| > 2\}} \right) \frac{K_1(|\lambda_0|(\hat{\lambda}_0 - \eta)) K_2(|\lambda_0|\eta)}{|\hat{\lambda}_0 - \eta|^{n - \kappa_1} |\eta|^{n - \kappa_2}} d\eta \\
&=: J_1(\lambda_0) + J_2(\lambda_0).
\end{aligned}$$

$$J_1(\lambda_0) \leq \int_{\{|\eta| < 2\}} \frac{\|K_1\|_\infty \|K_2\|_\infty d\eta}{|\widehat{\lambda}_0 - \eta|^{n-\kappa_1} |\eta|^{n-\kappa_2}} = \int_{\{|\eta| < 2\}} \frac{\|K_1\|_\infty \|K_2\|_\infty d\eta}{|\widehat{\mathbf{x}} - \eta|^{n-\kappa_1} |\eta|^{n-\kappa_2}} < \infty, \quad (4.48)$$

where the last equality holds for any unit vector $\widehat{\mathbf{x}}$.

$$J_2(\lambda_0) = \left(\int_{\{2 < |\eta| < 2(2 + \frac{1}{|\lambda_0|})\}} + \int_{\{|\eta| > 2(2 + \frac{1}{|\lambda_0|})\}} \right) \frac{K_1(|\lambda_0|(\widehat{\lambda}_0 - \eta)) K_2(|\lambda_0|\eta)}{|\widehat{\lambda}_0 - \eta|^{n-\kappa_1} |\eta|^{n-\kappa_2}} d\eta =: J_{2,1}(\lambda_0) + J_{2,2}(\lambda_0).$$

Because $|\widehat{\lambda}_0 - \eta| \geq |\eta| - 1$

$$\begin{aligned} J_{2,1}(\lambda_0) &\leq \int_{\{2 < |\eta| < 2(2 + \frac{1}{|\lambda_0|})\}} \frac{\|K_1\|_\infty \|K_2\|_\infty d\eta}{(|\eta| - 1)^{n-\kappa_1} |\eta_2|^{n-\kappa_2}} = c_n \int_2^{2(2 + \frac{1}{|\lambda_0|})} \frac{\|K_1\|_\infty \|K_2\|_\infty dr}{(r - 1)^{n-\kappa_1} r^{1-\kappa_2}} \\ &= c_n \|K_1\|_\infty \|K_2\|_\infty \int_2^{2(2 + \frac{1}{|\lambda_0|})} \frac{dr}{(1 - \frac{1}{r})^{n-\kappa_1} r} \leq 2^{n-\kappa_1} c_n \|K_1\|_\infty \|K_2\|_\infty \ln(2 + \frac{1}{|\lambda_0|}). \end{aligned} \quad (4.49)$$

Changing from variable η to $\frac{\tau}{|\lambda_0|}$ and using the inequality $|\lambda_0 - \tau| \geq |\tau| - |\lambda_0| \geq 2(1 + 2|\lambda_0|) - |\lambda_0| \geq 2$ for $|\tau| > 2(1 + |\lambda_0|)$,

$$J_{2,2}(\lambda_0) = \int_{\{|\tau| > 2(1+2|\lambda_0|)\}} \frac{K_1(\lambda_0 - \tau) K_2(\eta)}{|\lambda_0 - \tau|^{n-\kappa_1} |\tau|^{n-\kappa_2}} d\tau \leq \frac{\|K_1\|_\infty}{2^{n-\kappa_1}} \int_{\{|\tau| > 2(1+2|\lambda_0|)\}} \frac{K_2(\tau)}{|\tau|^{n-\kappa_2}} d\eta \leq \frac{\|K_1\|_\infty}{2^{n-\kappa_1}}.$$

The last estimation, together with (4.48) and (4.49), implies that there exist bounded and positive functions $\widetilde{B}(\lambda)$ and $C(\lambda)$ such that $g(\lambda) = \widetilde{B}(\lambda) \ln(2 + \frac{1}{|\lambda|}) + C(\lambda) = B(\lambda) \ln(2 + \frac{1}{|\lambda|})$, where $B(\lambda) = \widetilde{B}(\lambda) + \frac{C(\lambda)}{\ln(2 + \frac{1}{|\lambda|})}$ is also a bounded function.

Case 3: $\kappa_1 + \kappa_2 > n$, $\kappa_1, \kappa_2 > 0$. Because $\kappa_1 + \kappa_2 > n$ implies that there exist $p, p' \in (1, \infty)$ such that $p(n - \kappa_1), p'(n - \kappa_2) < n$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For any $\lambda \in \mathbb{R}^n$, by Hölder's inequality, $g(\lambda) \leq \|f_1\|_p \|f_2\|_{p'}$. Meanwhile, it also implies the continuity of g as follows:

$$|g(\lambda) - g(\lambda_0)| = \left| \int_{\mathbb{R}^n} (f_1(\lambda - \eta) - f_1(\lambda_0 - \eta)) f_2(\eta) d\eta \right| \leq \|f_1(\lambda - \cdot) - f_1(\lambda_0 - \cdot)\|_p \|f_2\|_{p'} \rightarrow 0$$

when $\lambda \rightarrow \lambda_0$ for any $\lambda_0 \in \mathbb{R}^n$.

Finally, by taking successive convolutions and using the result of Case 1, for any $k_1 > k_2 > n/\kappa$, $f^{*k_1}(\lambda)$ and $f^{*k_2}(\lambda)$ are bounded functions, which implies

$$f^{*k_1}(\lambda) = \int_{\mathbb{R}^n} f^{*k_2}(\lambda - \eta) f^{*(k_1-k_2)}(\eta) d\eta \leq \|f^{*k_2}\|_\infty \int_{\mathbb{R}^n} f^{*(k_1-k_2)}(\eta) d\eta = \|f^{*k_2}\|_\infty.$$

□

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